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ON CONCERNING COMPACT KÜRSCHAK FIELDS

A DISSERTATION

SUBMITTED TO THE FACULTY OF THE OGDEN GRADUATE SCHOOL OF SCIENCE IN
CANDIDACY FOR THE DEGREE OF DOCTOR OF PHILOSOPHY
DEPARTMENT OF MATHEMATICS

BY

VISHNU DATTATREYA GOKHALE

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Reprinted from
THE AMERICAN JOURNAL OF MATHEMATICS
Vol. XLIV, No. 4, October, 1922

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CONCERNING COMPACT KÜRSCHÁK FIELDS.

BY VISHNU DATTATREYA GOKHALE.

CONTENTS.

I. INTRODUCTION	298
1. Definition of a field, characteristic	298, 299
2. Algebraic closure, algebraic extensions	299
3. Definition of a Kürschák field	300
4. Perfection, Kürschák's results	300
II. COMPACTNESS AND ALGEBRAIC EXTENSIONS	302
1, 2. Compactness, compactness and perfection	302
3. Compactness and algebraic closure	303
4, 5. Hensel-Kürschák fields, Ostrowski's results, applications	304, 306
6. Compactness and adjunction of algebraic elements	306
III. THE HENSEL-KÜRSCHÁK FIELD $\mathfrak{K}_{\mathfrak{F}}$	308
1, 2. The Hensel-Kürschák field $\mathfrak{K}_{\mathfrak{F}}$	308, 310
3, 4. Perfection and compactness of $\mathfrak{K}_{\mathfrak{F}}$	310, 311
5. Subfield $\mathfrak{H}(\mathfrak{F}, x)$ of $\mathfrak{K}_{\mathfrak{F}}$	312
6. Adjunction of elements algebraic with respect to $\mathfrak{F}$	313
IV. THE PROPERTIES p, A, P, cpt	314
1-5. Complete existential theory of these four properties	314

Abstract.—A Kürschák field is a field with a modulus ("bewertete Körper), in the sense defined by Kürschák in his memoir, "Ueber Limesbildung und die allgemeine Körpertheorie" (Crelle, vol. 142, 1913). This modulus plays, in the general field, essentially the same role as the absolute value in the fields of classical analysis, viz., real number system, complex number system, etc. Kürschák proves that every Kürschák field determines (in the sense of isomorphism) a definite algebraically closed and perfect Kürschák superfield, the smallest such superfield, being (in the sense of isomorphism) a subfield of every such superfield. This definite superfield is the (smallest) algebraically closed and perfect extension of the original Kürschák field.

In the present paper the author sets up the notion *compactness*. This notion is analogous to M. Fréchet's compactness and to the J-compactness in E. H. Moore's General Analysis. It is a generalization of the following property in the point set theory: Every infinite set of points in a bounded domain has at least one condensation point. He then studies the properties of algebraically closed and compact fields, and compactness under the adjunction of algebraic elements. Using Ostrowski's results he proves the theorem that the smallest algebraically closed extension of a compact field is compact if, and only if, it can be obtained by adjoining a single algebraic element. The last part of the paper develops a complete existential theory of the four properties: (1) of characteristic other than zero, (2) algebraic closure, (3) perfection, and (4) compactness. Out of the $2^4 = 16$ possibilities 11 are shown to be existent and the remaining 5 non-existent.

I. INTRODUCTION.

1.1. **Field.**—Following Moore,* H. Weber,† and Steinitz,‡ a field is defined in the following manner: §

$$D \ 1.1 \quad \mathfrak{F} \equiv (\mathfrak{P} ; + \text{ on } \mathfrak{P} \text{ to } \mathfrak{P}, 1.2.5.6 ; \times \text{ on } \mathfrak{P} \text{ to } \mathfrak{P}, 3.4.5.7)$$

where $\mathfrak{P} \equiv [p]$ is a general class of elements $p(p_1, p_2, \text{etc.})$; $+$, $\times$ are single-valued functions, $+$ (p_1, p_2) is a definite element p denoted by $p_1 + p_2$, etc.; and the properties numbered 1–7 are:

$$\left. \begin{array}{l} (1) \quad p_1 + (p_2 + p_3) = (p_1 + p_2) + p_3 \\ (2) \quad p_1 + p_2 = p_2 + p_1 \\ (3) \quad (p_1 p_2) p_3 = p_1 (p_2 p_3) \\ (4) \quad p_1 p_2 = p_2 p_1 \\ (5) \quad p_1 (p_2 + p_3) = p_1 p_2 + p_1 p_3 \end{array} \right\} (p_1, p_2, p_3)$$

Here (p_1, p_2, p_3) indicates that the equations (1–5) hold for every choice of the elements $p_1 p_2 p_3$ of the class $\mathfrak{P}$.

$$(6) \quad p_1 \cdot p_2 \cdot \cdot \cdot \exists 1 \ p \ \ni \ p_1 + p = p_2$$

Here the meaning of the logical signs $\cdot \cdot \cdot$, $\exists$, 1 , and $\ni$ will be clear from the following reading of the property:—For every p_1 and p_2 , there exists uniquely an element p such that $p_1 + p = p_2$.

Hence,

$$(6_0) \quad \exists 1 \ z^{\mathfrak{P}} : \ni : p : \cdot \cdot \cdot p + z = p \cdot p z = z.$$

Here the notation $z^{\mathfrak{P}}$ reads “ z belongs to the class $\mathfrak{P}$.” This unique element z is called the zero element of the field.

$$(7) \quad \exists \ p \neq z : p_1 \neq z \cdot p_2 \cdot \cdot \cdot \exists 1 \ p \ \ni \ p_1 p = p_2,$$

to be read “There exists an element p different from the element z (of 6_0), and for every such element p_1 and every element p_2 there exists uniquely an element p such that $p_1 p = p_2$.”

* E. H. Moore, “A Doubly Infinite System of Simple Groups” (Chicago Congress, 1893, pp. 208–242), p. 210. This paper will be referred to as M.

† H. Weber, “Die allgemeinen Grundlagen der Galoischen Gleichungstheorie” (*Mathematische Annalen*, 43, 1893, pp. 521–549), p. 526; also “Algebra” (II edition, 1898), Vol. I, p. 492.

‡ Steinitz, “Algebraische Theorie der Körper” (*Crelle*, 137, 1910, pp. 167–309), p. 172. This paper will be referred to as S.

§ For postulational definitions of a field see:

L. E. Dickson, “Definitions of a Group and a Field by Independent Postulates” (*Trans. A. M. S.*, Vol. 6, 1905, pp. 198–204), p. 202.

E. V. Huntington, “Note on the Definitions of Abstract Groups and Fields by Sets of Independent Postulates” (*Trans. A. M. S.*, Vol. 6, pp. 181–197), pp. 186, 191. This paper also contains a bibliography.

This property can be easily seen to be equivalent to the following:

$$(7') \quad \exists p \neq z : \exists u^{\mathfrak{B}} \ni [p \cdot) \cdot pu = p : p \neq z \cdot) \cdot \exists p' \in pp' = u].$$

Also, we have

$$(7_0) \quad \exists 1 u^{\mathfrak{B}} \ni p \cdot) \cdot pu = \overline{p} : p_1 \neq z \cdot p_2 p_1 = z :) : p_2 = z.$$

This element u is called the *unit element* of the field, and the element (which can be easily seen to be unique) p' associated with p in the second part of $7'$ is called the *reciprocal* of p .

Hereafter we denote the elements p of the class $\mathfrak{B}$ of a field not by p but by f (f_1, f_2 , etc.).

1.11. **Characteristic.**—A field of *characteristic* p , in notation $\mathfrak{F}^p$ is defined in the following manner:*

$$D 1.2 \quad \mathfrak{F}^p : \equiv : \mathfrak{F} \cdot \ni \exists n \ni nu = z \cdot p \equiv \text{the smallest such } n.$$

By using the properties of the field it can be proved that in this case,

$$(1) \quad p \text{ is a prime.}$$

$$(2) \quad f \neq z :) : nf = z \cdot \sim \cdot n = \text{a multiple of } p.$$

On the other hand, for every positive integer m and a prime p , there exists† one and only one finite field, the so-called Galois field $[p^m]$, with characteristic p .

Fields without any such characteristic are said to be of characteristic zero, in notation $\mathfrak{F}^0$. Henceforth in the notation $\mathfrak{F}^p$ we shall understand p to be an indefinite prime number. Thus every field is of the type $\mathfrak{F}^0$ or $\mathfrak{F}^p$.

1.2. **Algebraic closure, algebraic extensions.**—An *algebraically closed* field $\mathfrak{F}$, in notation $\mathfrak{F}^A$, is thus defined:‡

$$D 1.3 \quad \mathfrak{F}^A : \equiv : \mathfrak{F} : \ni :$$

$$n \cdot (f_0, f_1, \dots, f_n) :) : \exists f \ni f_0 f^n + f_1 f^{n-1} + \dots + f_n = z.$$

An element j *algebraic with respect to* $\mathfrak{F}$, in notation $j^{\text{alg } \mathfrak{F}}$ is defined as:§

$$D 1.4 \quad j^{\text{alg } \mathfrak{F}} : \equiv : (j, \mathfrak{F}) : \ni : \exists \cdot^{\cdot 1} n \cdot (f_1, \dots, f_n) \cdot \ni \cdot$$

$$(x^n + f_1 x^{n-1} + \dots + f_n)^{\text{irreducible in } \mathfrak{F}} \cdot j^n + \dots + f_n = z.$$

This unique integer n is called the *order* of j . Here x is an indeterminate, and j belongs to a field $\mathfrak{F}'$ containing $\mathfrak{F}$ as a subfield, in notation, $\mathfrak{F}' \supset \mathfrak{F}$.

Steinitz has shown|| how to extend a field by the adjunction of an

* S, p. 181. Cf. also: J. König:—Einleitung in die allgemeine Theorie der algebraischen Grössen (Leipzig, 1913), p. 408. König uses the terms "orthoid" and "pseudoorthoid" for fields of characteristic zero and p respectively.

† M, p. 211.

‡ S, p. 260.

§ S, p. 183.

|| S. p. 197.

algebraic element j . This extension of a field $\mathfrak{F}$, in notation $\mathfrak{F}(j)$, is the extension in the sense that it is the smallest and unique (in the sense of isomorphism). He also shows how to get the extension which reduces completely a given polynomial φ irreducible in $\mathfrak{F}$. Steinitz also proves:*

Thm 1.1 $\mathfrak{F} :) : \exists \mathfrak{F}'^{\supset \mathfrak{F}} : \ni : \mathfrak{F}'^A \cdot \mathfrak{F}''^{A \cdot \supset \mathfrak{F}} \cdot \mathfrak{F}'' \supset \mathfrak{F}'$.

This field $\mathfrak{F}'$, unique in the sense of isomorphism, we denote by $\mathfrak{F}_A$: the (smallest) extension of $\mathfrak{F}$ having the property A .

Every field $\mathfrak{F}$ has one and only one prime subfield $\dagger \mathfrak{P}_0$; $\mathfrak{P}_0$ is isomorphic with the rational number system or the integral number system taken modulo p , according as $\mathfrak{F}$ is of characteristic 0 or p . By *absolute algebraic field*, in notation $\mathfrak{P}$, we mean the algebraically closed extension of such a prime field $\mathfrak{P}_0 \ddagger$

A field $\mathfrak{F}'$, algebraic extension of $\mathfrak{F}$, in notation $\mathfrak{F}'^{\text{alg } \mathfrak{F}}$, is thus defined as: §

D 1.5 $\mathfrak{F}'^{\text{alg } \mathfrak{F}} : \equiv : (\mathfrak{F}', \mathfrak{F}) : \ni : \mathfrak{F}' \supset \mathfrak{F} \cdot f'^{-\mathfrak{F}} \cdot) \cdot f'^{\text{alg } \mathfrak{F}}$.

Note: The negative sign denotes the absence of the property. Thus the last implication should be read: *If f' does not belong to $\mathfrak{F}$, it is algebraic with respect to $\mathfrak{F}$.*

1.3. Kürschák Field.—A Kürschák field, in notation $\mathfrak{K}$, is thus defined: ||

D 1.6 $\mathfrak{K} \equiv (\mathfrak{F} ; || \quad || \text{ on } \mathfrak{F} \text{ to } \mathfrak{K}^{\text{real.} \geq 0} \cdot 1, 2, 3, 4),$

where $\mathfrak{F}$ denotes a field, $\mathfrak{K}^{\text{real.} \geq 0}$ the class of all real non-negative numbers, and the properties 1-4 of the single-valued function $|| \quad ||$ (which will hereafter be called the *modulus*). are:—

- $$\left. \begin{aligned} (1) \quad & f = z \cdot \sim \cdot || f || = 0 \\ (2) \quad & || f_1 f_2 || = || f_1 || || f_2 || \quad \therefore \quad || u || = 1 \\ (3) \quad & || f_1 + f_2 || \leq || f_1 || + || f_2 || \\ (4) \quad & \exists f : \ni : || f || \neq 0 \cdot || f || \neq 1 \end{aligned} \right\} (f_1, f_2).$$

Hereafter we designate the elements of $\mathfrak{K}$ not by f but by $k(k_1, k_2, \text{etc.})$.

1.4. Limit, Perfect Extension.—A limit k of a sequence $k_1, k_2, \dots$ in notation $\{k_n\}$, is thus defined: ¶

D 1.7 $L_n k_n = k : \equiv : (k, \{k_n\}) \ni L_n || k - k_n || = 0.$ *

* S, p. 287.

† S, p. 180.

‡ S, p. 199.

§ S, p. 198.

|| J. Kürschák, "Über Limesbildung und allgemeine Körpertheorie" (*Crelle*, 142, 1913, pp. 211-233), p. 211. This paper will be referred to as K.

¶ K, p. 222.

As an immediate consequence of the definition,

$$\text{Thm 1.2} \quad L_n k_n = k \cdot L_n k_n = k' :) : k = k' \cdot \|k\| = \|k'\| = L_n \|k_n\|.$$

The usual properties of limits, for instance: if a sequence has a limit, every subsequence has the same limit and conversely; the sum of limits equals the limit of the sum; etc., follow in the usual manner.

A Cauchy sequence $\{k_n\}$, or $\{k_n\}$ satisfying the Cauchy condition: in notation $\{k_n\}^{c.c.}$: is:

$$D 1.8 \quad \{k_n\}^{c.c.} \equiv : \{k_n\} : \vartheta : e :) : \exists n_e \cdot \vartheta \cdot n > n_e \cdot) \cdot \|k_n - k_{n_e}\| < e.$$

Here, as further, e is a real positive number. As an immediate consequence, we have

$$\text{Thm 1.3} \quad \exists L_n k_n \cdot) \cdot \{k_n\}^{c.c.}$$

The converse however is not true; e.g., in the field of rational numbers with the absolute value as modulus, not all Cauchy sequences have rational limits. A field $\mathfrak{R}_0$ or a subclass $\mathfrak{R}_0$ (not necessarily a field) of a field $\mathfrak{R}$ for which the converse holds is called *perfect*: in notation $\mathfrak{R}_0^P$. Thus the definition of perfection is: *

$$D 1.9 \quad \mathfrak{R}_0^P \equiv : \mathfrak{R}_0^{\mathfrak{R} \subset} : \vartheta : \{k_{0n}\}^{c.c.} \cdot) : \exists k_0 \vartheta L_n k_{0n} = k_0;$$

in this definition (as also further, where necessary) we denote the elements of $\mathfrak{R}_0$ by $k_0(k_{01}, k_{02}, \text{etc.})$.

By a method analogous to G. Cantor's in building up the real number system from the rational numbers, Kürschák has shown† how to extend a Kürschák field so as to make it perfect. The smallest such extension (unique in the sense of isomorphism) of a $\mathfrak{R}$ we designate notationally by $\mathfrak{R}_A$: the (smallest) extension of $\mathfrak{R}$ having the property P .

Given a $\mathfrak{R}^P$ and $j^{\text{alg } \mathfrak{R}}$, Kürschák defines $\|j\|$ so that $\mathfrak{R}(j)$ is a Kürschák field: ‡ in notation $\mathfrak{R}(j)^K$. Defining in $\mathfrak{R}_A$ the modulus of every element algebraic with respect to $\mathfrak{R}$ in the same manner, we have $\mathfrak{R}_A^K$. Making this perfect, we get $\mathfrak{R}_{AP}^P$, in the sense $(\mathfrak{R}_A)_P$. Kürschák then shows§ that $\mathfrak{R}_{AP}$ is algebraically closed. If we start with any Kürschák field $\mathfrak{R}$, we

* K, p. 228.

† K., p. 228.

‡ K, p. 245. Kürschák defines $\|j\|$ to be $\|f_n\|^{1/n}$, where f_n is the f_n in D 1.4. A. Ostrowski in his "Über sogenannte perfecte Körper" (*Crelle*, 147, 1917, pp. 191-204), p. 196, and "Über einige Lösungen der Functionalgleichung $\varphi(xy) = \varphi(x)\varphi(y)$ " (*Acta Mathematica*, 41, 1918, p. 271-284), p. 280, has shown that this is the only possible definition of $\|j\|$.

These papers will be referred to as O_2, O_3 respectively.

§ K, p. 251.

must first make it perfect and then follow this process. Thus:

$$\mathfrak{K} \rightarrow \mathfrak{K}_P^P \rightarrow \mathfrak{K}_{PA}^A \rightarrow^* \mathfrak{K}_{PA}^{A.P.}$$

Kürschák's final theorem is thus: †

$$\text{Thm 1.4} \quad \mathfrak{K} :) : \mathfrak{K}_{PA}^{A.P.} \mathfrak{K} \cdot \mathfrak{K}'^{A.P.} \supset \mathfrak{K}' \supset \mathfrak{K}_{PA}^P$$

This perfect and algebraically closed extension of $\mathfrak{K}$ we shall denote by $\mathfrak{K}_{AP}$ or $\mathfrak{K}_{PA}$.

II. COMPACTNESS AND ALGEBRAIC EXTENSIONS.

2.1. Compactness.—In classical point set theory we have the theorem that every infinite set of points in a bounded domain has at least one condensation point. A similar property for a general class for which the limit function exists is defined by Fréchet.‡ Classes having this property are said to be, according to his nomenclature, compact. The following definition of compactness in the case of a Kürschák field is naturally suggested as analogous to this definition of Fréchet and the definition of J -compactness in Moore's theory.§

A subset (not necessarily a field) of a Kürschák field is said to be compact if every infinite sequence of elements of the set such that the (smallest) upper bound of the modulus is finite, contains at least one Cauchy subsequence with its limit element in the set.

In notation:

$$D \ 2.1 \quad \mathfrak{K}_0^{cpt} : \equiv : \mathfrak{K}^{\subset \mathfrak{K}} : \ni : \{k_{0n}\} \ni \overline{B} \parallel k_{0n} \parallel < \infty \\ \cdot) \cdot \exists (k_0, \{n_m\}^{pr. mon. inc.}) \ni \lim_n k_{0n_m} = k_0.$$

Here as in $D \ 1.9$ $\mathfrak{K}_0$ is a subset not necessarily a field of a Kürschák field $\mathfrak{K}$; and $\{n_m\}$ is a *properly monotonic increasing sequence* of positive integers n . This property denoted notationally in the above definition as $pr. mon. inc.$ is thus defined:

$$D \ 2.11 \quad \{n_m\}^{pr. mon. inc.} : \equiv : \{n_m\} : \ni : m_1 > m_2 \cdot) \cdot n_{m_1} > n_{m_2}.$$

2.2. Compactness and Perfection.—We now prove that:

$$\text{Thm 2.1} \quad \mathfrak{K}_0^{cpt} \cdot) \cdot \mathfrak{K}_0^P.$$

* In his theses, "Über einige Fragen der allgemeinen Körpertheorie" (*Crelle*, 143, 1913, pp. 255–284), p. 284, A. Ostrowski finds under what conditions this step is necessary. He also shows (p. 260) that the algebraically closed extension of Hensel's p -adic numbers is not perfect. This paper will be referred to as O_1 .

† K, p. 251.

‡ M. Fréchet, "Sur quelques points du calcul fonctionnel" (*Rendiconti del Circ. Math. d. Palermo*, 22, 1906), p. 6.

§ E. H. Moore, "Lectures on Matrices in General Analysis" (University of Chicago, 1919–1920).

Proof: Consider a Cauchy sequence $\{k_{0n}\}$. Since it is a Cauchy sequence,

$$e :) : \exists n_e \ni n > n_e \cdot) \cdot \| k_{0n} - k_{0n_e} \| < e.$$

Therefore for such an n , $\| k_{0n} \|$ lies between $|\| k_{0n_e} \| - e|$ and $\| k_{0n_e} \| + e$. Hence $\overline{B}_n \| k_{0n} \| < \infty$. Therefore by compactness we have a subsequence having a limit in $\mathfrak{K}_0$. Hence the original sequence has the same limit in $\mathfrak{K}_0$. Hence the theorem.

Note: The converse of this theorem, however, is not true. See *Thm* 2.3 below. Compare also theorems 3.3 and 3.4 in the next section.

2.3. Compactness and Algebraic Closure.—

Thm 2.2 $\mathfrak{K}^{A, \text{cpt}} \cdot a \cong 0 :) : \exists k \ni \| k \| = a.$

Proof: The theorem is obvious when $a = 0$ or 1, the corresponding k being z and u respectively. When a is neither the proof is as follows:

By *D* 1.6 property 4, there exists an element whose modulus is neither 0 nor 1. Let this element be denoted by k_0 and let its modulus be ξ .

By the theory of the real number system $\exists \{r_n\} \ni \lim_n \xi^{r_n} = a$, where $\{r_n\}$ is a sequence of ordinary rationals r . Hence by $\mathfrak{K}^A$,

$$\exists \{k_0^{r_n}\} \cdot \overline{B}_n \| k^{r_n} \| < \infty.$$

Therefore by $\mathfrak{K}^{\text{cpt}}$,

$$\exists (k, \{n_m\}^{\text{pr. mon. line.}}) \ni \lim_m k^{r_{n_m}} = k$$

and $\| k \| = \lim_m \| k_0^{r_{n_m}} \| = \lim_m \xi^{r_{n_m}} = a.$

Using Moore's results,* Steinitz proves:†

Thm 2.3 $f^{1/p} \neq z :) : \exists m \ni f^{p^{m-1}} - u = z.$

We shall use this theorem to prove:

Thm 2.4 $\mathfrak{K}^{A, \text{cpt}} \cdot) \cdot \mathfrak{K}^0.$

Proof: We shall give a direct proof of a contrapositive of the theorem, viz.,

$$\mathfrak{K}^{p^A} \cdot) \cdot \mathfrak{K}^{-\text{cpt}}.$$

Every $\mathfrak{K}^{p^A}$ contains $\mathfrak{B}^p$, the absolute algebraic field characteristic p . Consider $\{k_n\}^{\text{distinct. } p^p}$. By theorem 2.3 we have:

$$n :) : \| k_n \| = 0 \quad \text{or} \quad \| k_n \|^q = 1,$$

where $q = p^m - 1$ in theorem 2.3. Hence $\overline{B}_n \| k_n \| = 1 \therefore < \infty$. But

* M, p. 220.

† S, p. 251.

this sequence cannot have a Cauchy subsequence; for, otherwise,

$$e :) : \exists (n_1, n_2)^{\text{distinct}} \ni \|k_{n_1} - k_{n_2}\| < 2e,$$

viz., n_1, n_2 each greater than n_e in D 1.8. But from theorem 2.3 and the fact that $n_1 \neq n_2 \cdot) \cdot \|k_{n_1}\| \neq \|k_{n_2}\|$, we have

$$n_1 \neq n_2 \cdot) \cdot \|k_{n_1} - k_{n_2}\| = 1.$$

Taking $e < 1/2$ we get a contradiction. Thus this particular sequence has no Cauchy subsequence. Hence the theorem.

We have further:

$$\text{Thm 2.5} \quad \mathfrak{K}^{\text{cpt}} \cdot) \cdot \exists n \ni \|nu\| \neq 1.$$

Proof: Here also we prove the contrapositive, viz.,

$$\mathfrak{K} \ni n \cdot) \cdot \|nu\| = 1 :) : \mathfrak{K}^{\text{cpt}}.$$

Since $\|nu\| = 1$ (n), we have the characteristic zero, and $\|ru\| = 1$ (r) where r is any ordinary rational different from zero. Consider $\{r_n\}^{\text{distinct}}$. We have $\bar{B}_n \|r_n u\| = 1 \therefore < \infty$.

Now the sequence $\{r_n u\}$ cannot have any Cauchy subsequence; for otherwise,

$$e :) : \exists (n_1, n_2)^{\text{distinct}} \ni \|r_{n_1} u - r_{n_2} u\| < 2e,$$

but since $n_1 \neq n_2 \cdot) \cdot r_{n_1} u - r_{n_2} u = (r_{n_1} - r_{n_2})u \neq z \cdot) \cdot \|r_{n_1} u - r_{n_2} u\| = 1$, taking $e < 1/2$ we get a contradiction. Hence the theorem.

We have also:

$$\text{Thm 2.6} \quad \mathfrak{K}^{\text{cpt}} \cdot) \cdot (\text{the first } n \ni \|nu\| \neq 1)^{\text{prime}}.$$

Proof: For otherwise if $n = n_1 n_2$, where $n_1 \leq n_2 < n$, $nu = n_1 n_2 u = (n_1 u)(n_2 u)$. Hence $\|nu\| = 1$ which contradicts the hypotheses.

2.4. Hensel-Kürschák Fields, Ostrowski's Results.—A Kürschák field where the property 3 of the modulus in D 1.6 is replaced by the stronger property:

$$(3') \quad \|f_1 + f_2\| \leq \text{greater of } (\|f_1\|, \|f_2\|) \quad (f_1, f_2)$$

is called a Hensel-Kürschák field: in notation $\mathfrak{S}$. Thus:

$$D \ 2.2 \quad \mathfrak{S} \equiv (\mathfrak{F}; \| \cdot \|_{\text{on } \mathfrak{S} \text{ to } \mathfrak{R}^{\text{real}} \cong 0.1.2.3'.4),$$

where the properties 1, 2, 4 are those in D 1.6, and $3'$ is defined above. Hereafter we denote the elements of $\mathfrak{S}$ by $h(h_1, h_2, \text{etc.})$; the class of all Hensel-Kürschák fields will be denoted by H , and the superscript H will denote the property of belonging to this class. These notations will be used only when it is necessary to emphasize the fact that the Kürschák field under consideration is a Hensel-Kürschák field.

The modulus of a Hensel-Kürschák field, which, when necessary, we shall call Hensel-Kürschák modulus, is what Ostrowski* calls a non-Archimedean modulus: in notation: $\| \|_{\text{non-arch}}$. He also proves:

$$\text{Thm} \dagger 2.7 \quad \| \|_{\text{non-arch}} : \sim : n \cdot) \cdot \| nu \| \leq 1.$$

$$\text{Thm} \ddagger 2.8 \quad \| h_1 \| > \| h_2 \| :) : \| h_1 + h_2 \| = \| h_1 \|.$$

A modulus which does not obey this stronger property 3' is called Archimedean: in notation: $\| \|_{\text{arch}}$.

A. Ostrowski has investigated real-valued solutions of the functional equations:

$$\left. \begin{aligned} \varphi(xy) &= \varphi(x)\varphi(y) \\ \varphi(x+y) &\leq \varphi(x) + \varphi(y) \end{aligned} \right\} (x, y)^{\text{rational}}.$$

His conclusions, modified for the Kürschák modulus by the fact that the modulus is non-negative, are:

$$\begin{aligned} \text{Thm} \S 2.9 \quad \Re^0 :) : \quad & (1) r \neq z \cdot) \cdot \| r \| = 1; \| z \| = 0 \\ & \text{or } (2) r \cdot) \cdot \| r \| = |r|^\rho \quad 0 < \rho \leq 1 \\ & \text{or } (3) r \cdot) \cdot \| r \| = c^{0(p, r)} \quad 0 < c < 1, p^{\text{prime}}. \end{aligned}$$

Here $0(p, r)$ is the order of r with respect to p , defined, following Hensel $\|$ as:

$$D 2.3 \quad \left[r = \frac{u}{v} p^\rho \ni (u, v)^{\text{prime to } p} :) : 0(p, r) \equiv \rho. \right]$$

In the above theorem, the modulus is Archimedean in case 2, non-Archimedean in the other two. For $n = 2$ in theorem 2.6, there does exist a field with Archimedean modulus, viz., $\Re^{\text{complex}}$, with the modulus as defined in the second part of theorem 2.9. If we denote this field by $\Re_p$, and define equivalence: in notation $\sim :$ as isomorphism between Kürschák fields which is preserved under the limit process, we have:

$$\text{Thm 2.10} \quad \rho_1 \cdot \rho_2 :) : \Re_{\rho_1} \sim \Re_{\rho_2} \quad \left(\begin{array}{l} 0 < \rho_1 \leq 1 \\ 0 < \rho_2 \leq 1 \end{array} \right).$$

Ostrowski proves: ¶

$$\text{Thm 2.11} \quad \Re^{-H \cdot)} \cdot \exists \rho \ni \Re \subset \Re_p$$

and finally,**

$$\text{Thm 2.12} \quad \Re^{-H.P.A. \cdot)} \cdot \Re \sim \Re^{\text{complex}}.$$

* *O*₃, p. 273.

† *O*₃, p. 273.

‡ *O*₃, p. 274.

§ *O*₃, p. 276.

¶ K. Hensel, "Theorie der algebraischen Zahlen" (Leipzig, 1903, Vol. I).

¶ *O*₃, p. 281.

** *O*₃, p. 282.

2.5. **Deductions from Ostrowski's Results.**—From theorem 2.7 the Archimedian or non-Archimedian character of the modulus depends upon the moduli of the elements of the prime subfield. Hence we have:

$$\text{Thm 2.13} \quad \mathfrak{K}' \supset \mathfrak{K} :) : \mathfrak{K}^{H(-H)} \cdot \sim \cdot \mathfrak{K}'^{H(-H)}.$$

Here the two parentheses go together.

From theorem 2.12 we have

$$\text{Thm 2.14} \quad \mathfrak{K}^{-H.P.A. \cdot}) \cdot \mathfrak{K}^{\text{cpt}}.$$

We shall now prove an important theorem to be used in the sequel: viz., *compactness is not extensionally attainable*.* To be more precise,

$$\text{Thm 2.15} \quad \mathfrak{K}^{-\text{cpt. } P \cdot}) \cdot - \exists \mathfrak{K}' \supset \mathfrak{K}^{\text{cpt}}.$$

Proof: Since the field is perfect every Cauchy sequence has a limit element in the field. Want of compactness, therefore, is due to the fact that there exists an infinite sequence which has no Cauchy subsequence. Now every extension preserves the moduli; this fact will therefore persist in every extension. Hence no extension will make the field compact.

We show in section 4 theorem 4.10₀ that $\mathfrak{S}_{(PA)}^{-\text{cpt}}$, where $\mathfrak{S}$ is the Hensel field of p -adic numbers. Since in this case compactness is not extensionally attainable (in virtue of theorem 2.15), we have, using theorems 2.9 and 2.4,

$$\text{Thm 2.16} \quad \mathfrak{K}^{A. \text{cpt}} : \sim : \mathfrak{K} \sim \mathfrak{K}^{\text{complex}}.$$

2.6. **Compactness and Adjunction of Algebraic Elements.**—An algebraic element of the first kind: in notation alg_1 : is thus defined: †

$$\text{D 2.4} \quad j^{\text{alg}_1} : \mathfrak{K} \equiv : (j, \mathfrak{K}) : \ni j^{\text{alg } \mathfrak{K}} : \varphi^{\text{irreducible in } \mathfrak{K}} \cdot \varphi(j) = z \cdot \exists \psi \ni \psi(j) \neq z \cdot \varphi = (\cdot - j)\psi.$$

Note: Here the dot in the last brackets stands for the argument of the function $[x - j | x]$: read “ $x - j$ as x varies.”

An algebraic element, which is not of the first kind is said to be of the second kind: in notation: alg_2 . Similar definitions hold for algebraic extensions of the first and second kind.

Ostrowski proves: ‡

$$\text{Thm 2.17} \quad \mathfrak{K}^{P \cdot j^{\text{alg}_1 \mathfrak{K}} \cdot}) \cdot \mathfrak{K}(j)^P.$$

We prove the analogous theorem:

$$\text{Thm 2.18} \quad \mathfrak{K}^{\text{cpt} \cdot j^{\text{alg}_1 \mathfrak{K}} \cdot}) \cdot \mathfrak{K}(j)^{\text{cpt}}.$$

* A property is said to be extensionally attainable when there exists an extension which has that property. Compare E. H. Moore, “Introduction to a Form of General Analysis,” Yale University Press, 1910, pp. 53, 54.

† S, p. 231.

‡ O₁, p. 275.

Proof: Let m be the order of j and μ an upper bound of the modulus of an infinite sequence $\{k_{n,1}j^{m-1} + \dots + k_{n,m} \mid n\}$ of elements of $\mathfrak{R}(j)$. We have to show that this sequence has at least one Cauchy subsequence with a limit element in $\mathfrak{R}(j)$.

Let us denote this sequence by $\{i_{0,n}\}$ and let the m conjugates of $i_{0,n}$ be denoted by $i_{1,n}, i_{2,n}, \dots, i_{m,n}$; $i_{1,n} \equiv i_{0,n}(n)$. Let the m conjugates of j be denoted by $j_1, j_2, \dots, j_m$; $j \equiv j_1$.

In the normal extension* of $\mathfrak{R}$ which contains $\mathfrak{R}(j)$ we have for every n the m linear equations:

$$\begin{aligned} k_{n,1}j_1^{m-1} + \dots + k_{n,m} &= i_{1,n} \\ k_{n,1}j_2^{m-1} + \dots + k_{n,m} &= i_{2,n} \\ &\vdots \\ k_{n,1}j_m^{m-1} + \dots + k_{n,m} &= i_{m,n} \end{aligned}$$

Since j is of the first kind, the determinant of coefficients involving j and its conjugates is not the zero element. Hence we can solve for the k 's and get $k_{n,r}$ (r, n) as linear functions of the i 's with coefficients independent of n . Also $\|i_{r,n}\| = \|i_{0,n}\|(n, r)$. Hence $\bar{B} \|k_{n,r}\| < \infty$ (r). Consider

now the sequence $\{k_{n,1}\}$; by $\mathfrak{R}^{\text{cpt}}$ and the result proved just now, this sequence has at least one Cauchy subsequence with a limit element in $\mathfrak{R}$. Take one such Cauchy subsequence and the corresponding subsequence of $\{i_{0,n}\}$, say $\{i'_n\}$. Denote the coefficients of the different powers of j in this sequence by prime letters. As above we can prove that $\{k'_{n,2}\}$ has a Cauchy subsequence. Take the corresponding subsequence of $\{i'_n\}$ which is itself a subsequence of the original sequence. Continuing this process, after a finite number of steps, viz., m steps, we get a subsequence of the original sequence such that the coefficient sequences are all Cauchy sequences. Let this final sequence be denoted by $\{i_n\}$. Then if

$$i_n \equiv k_{n,1}j^{m-1} + \dots + k_{n,m} \quad (n),$$

we have:

$$n_1 \cdot n_2 :) : \|i_{n_1} - i_{n_2}\| \leq gr(\|k_{n_1,1} - k_{n_2,1}\|, \dots, \|k_{n_1,m} - k_{n_2,m}\|) \\ gr(\|j\|^{m-1}, \dots, \|j\|, 1),$$

where gr denotes "the greater of." Hence this subsequence is a Cauchy sequence. Using theorems 2.1 and 2.17 we see that the limit element is in $\mathfrak{R}(j)$. Hence the theorem.

An algebraic extension is said to be *finite* with respect to the original field: in notation $\mathfrak{A}_n$: when it can be obtained by adjoining a finite number

* S., p. 207.

of algebraic elements to the original field:*

$$D\ 2.5 \quad \mathfrak{K}^{\text{alg}} : \equiv : (\mathfrak{K}', \mathfrak{K}) : \ni \exists n \cdot (j_1, j_2, \dots, j_n)^{\text{alg}} \ni \mathfrak{K}' = \mathfrak{K}(j_1, \dots, j_n).$$

Steinitz proves: †

$$Thm\ 2.19 \quad \mathfrak{K} \cdot (j_1, \dots, j_n)^{\text{alg}} :) : \ni j^{\text{alg}} \ni \mathfrak{K}(j) = \mathfrak{K}(j_1, \dots, j_n).$$

Ostrowski proves: ‡

$$Thm\ 2.20 \quad \mathfrak{K}^P :) : \mathfrak{K}_A^{\text{an}} \cdot) \cdot \mathfrak{K}_A^{\text{alg}}.$$

Using theorem 2.16, he then proves§ a theorem, which, in view of theorem 2.19, we can state as:

$$Thm\ 2.21 \quad \mathfrak{K}^P :) : \mathfrak{K}_A^P \cdot \sim \cdot \ni j^{\text{alg}} \ni \mathfrak{K}(j) = \mathfrak{K}_A.$$

We prove the analogous theorem:

$$Thm\ 2.22 \quad \mathfrak{K}^{\text{cpt}} :) : \mathfrak{K}_A^{\text{cpt}} \cdot \sim \cdot \ni j^{\text{alg}} \ni \mathfrak{K}(j) = \mathfrak{K}_A.$$

Proof: Since compactness implies perfection, from 2.21 we have

$$\mathfrak{K}_A^{\text{cpt}} \cdot) \cdot \ni j^{\text{alg}} \ni \mathfrak{K}(j) = \mathfrak{K}_A.$$

The other part of the theorem follows from theorems 2.20 and 2.18.

III. THE HENSEL-KÜRSCHÁK FIELD $\mathfrak{K}_{\mathfrak{S}}$.

3.1. **The Field $\mathfrak{K}_{\mathfrak{S}}$.**—Given a field $\mathfrak{F}$ we build a system: ||

$$D\ 3.1 \quad \mathfrak{K}_{\mathfrak{S}} \equiv [\text{all } \varphi^{\text{on } \mathfrak{S} \text{ to } \mathfrak{S}} : \ni \exists i_{\varphi} \ni i < i_{\varphi} \cdot) \cdot \varphi(i) = z].$$

Here $\mathfrak{S} = [i]$ is the class of all integers i .

Addition of elements of the system is the usual addition of functions:

$$D\ 3.2 \quad \varphi_1 + \varphi_2 \equiv (\varphi_1(i) + \varphi_2(i) \mid i) \quad (\varphi_1, \varphi_2).$$

Multiplication is thus defined: ¶

$$D\ 3.3 \quad \varphi_1 \varphi_2 \equiv \left(\sum_{i_1, i_2 \mid i_1 + i_2 = i} \varphi_1(i_1) \varphi_2(i_2) \mid i \right) \quad (\varphi_1, \varphi_2).$$

* S, p. 199.

† S, p. 220.

‡ O₁, p. 281.

§ O₁, p. 284.

|| In this definition it is obvious that when φ is not the zero function the i_{φ} 's in the definition have a maximum. We denote this maximum by $\nu(\varphi)$ or ν_{φ} or more simply by ν when it is obvious to which element it belongs. Also in this case $\varphi(\nu) \neq z$. In case φ is the zero function such a maximum does not exist; we denote this symbolically by regarding ν as ∞ .

¶ It is to be noticed that the summation in this definition is finitely non-zero, and hence no questions like convergence, etc., are involved; for by D 3.1 both i_1 and i_2 have finite lower bounds for which the functional values of φ_1, φ_2 respectively are non-zero, and since $i_1 + i_2 = i$ the upper bounds too will be finite and for values greater than these the products of the functional values will be zero.

We shall hereafter designate the system in D 3.1 with addition and multiplication as defined in 3.2 and 3.3 by $\mathfrak{K}_{\delta}$. We also introduce a notation for a special set of elements of $\mathfrak{K}_{\delta}$.

D 3.4 $\delta_i \equiv (u \text{ at } i \text{ and } z \text{ elsewhere}) \quad (i).$

We now prove the theorem that this system is a field:

Thm 3.1 $\mathfrak{F} \cdot) \cdot \mathfrak{K}_{\delta}^F.$

Proof: The properties 1-6 and hence δ_0 in D 1.1 are easily seen to be satisfied from the fact that $\mathfrak{F}$ is a field; the zero element is in this case the zero function $\varphi(i) = z(i)$. As for the property 7 we shall prove its equivalent 7'. The first part of this property is obvious; also the element δ_0 satisfies the first condition for the unit element, and in fact also the second condition, that is,

$$\varphi \neq z^{\mathfrak{K}_{\delta}} \cdot) \cdot \exists \phi' \ni \varphi \phi' = \delta_0.$$

Consider the ν belonging to this φ (footnote 3.1); then an effective function φ' is the function φ' with $\varphi'(i) \equiv z$ for $i < -\nu$, $\varphi'(-\nu) \equiv u/\varphi(\nu)$, and $\varphi'(i)$ for $i = -\nu + s$ ($s > 0$) as the elements of $\mathfrak{F}$ obtained by the sequential solutions for $s = 1, 2, \dots$, of the equations:

$$\sum_{i_1, i_2 | i_1 + i_2 = s} \varphi(\nu + i_1) \varphi'(-\nu + i_2) = z \quad (s > 0).$$

Note 1: If we denote symbolically δ_i by x^i , the elements of the field $\mathfrak{K}_{\delta}$ can be symbolically expressed as infinite series $\sum_i f_i x^i$, where $f_i \equiv \varphi(i)$ (i). Addition and multiplication can then be regarded as the corresponding formal operations on the infinite series. We shall, therefore, hereafter take these infinite series as the elements of $\mathfrak{K}_{\delta}$ and operate with them since this procedure is formally more convenient. We shall sometimes denote the range of summation of the index i as from ν to ∞ , ν being the integer defined in footnote 3.1.

Note 2: We now identify (in the sense of isomorphism) this field with the field $\mathfrak{F}(x)$ of Steinitz, where x is transcendental* with respect to $\mathfrak{F}$. Putting the various powers of x in one field into correspondence with the same powers of x in the other, and the elements of $\mathfrak{F}$ with the same elements of $\mathfrak{F}$ in the other, we see that $\mathfrak{K}_{\delta} \supset \mathfrak{F}[x]$, where $\mathfrak{F}[x]$ is the integral domain obtained by adjoining the transcendental x . Since the field $\mathfrak{F}(x)$ is obtained from this integral domain by the process of building up of quotients,† we see that $\mathfrak{K}_{\delta} \supset \mathfrak{F}(x)$. On the other hand it is obvious that $\mathfrak{K}_{\delta} \subset \mathfrak{F}(x)$. Hence the result required.

* S, p. 184.

† S, p. 178.

We shall therefore now speak of this process of forming $\mathfrak{K}_\xi$ from $\mathfrak{F}$ as the adjunction of a transcendental element x to the field $\mathfrak{F}$.

3.2. The Hensel-Kürschák Field $\mathfrak{K}_\xi$.—Let ξ be a real number $0 < \xi < 1$. We now define the modulus for the field $\mathfrak{K}_\xi$:

$$D \ 3.5 \quad ||z^{\mathfrak{K}_\xi}|| \equiv 0; \quad \varphi \neq z^{\mathfrak{K}_\xi} \cdot \varphi \Rightarrow ||\varphi|| \equiv \xi^{\nu\varphi}.$$

The modular properties 1 and 4 of $D \ 1.6$ are obvious. The property 2 is also obvious in case one of the factors is the zero function. In the other cases,

$$\nu(\varphi_1\varphi_2) = \nu(\varphi_1) + \nu(\varphi_2) \quad (\varphi_1, \varphi_2).$$

Hence

$$||\varphi_1\varphi_2|| = \xi^{\nu(\varphi_1\varphi_2)} = \xi^{\nu(\varphi_1) + \nu(\varphi_2)} = \xi^{\nu(\varphi_1)}\xi^{\nu(\varphi_2)} = ||\varphi_1|| ||\varphi_2||.$$

Thus 2 is proved completely.

Again, for every φ_1 and φ_2 , $\nu(\varphi_1 + \varphi_2)$ = the first i for which $\varphi_1(i) + \varphi_2(i)$ is not zero (if there is any such, otherwise $\varphi_1 + \varphi_2 = z$ and the property 3' is obvious); thus

$$\nu(\varphi_1 + \varphi_2) \geq \text{the smaller of } \nu(\varphi_1), \nu(\varphi_2).$$

In this case since $0 < \xi < 1$, we have:

$\varphi_1 \cdot \varphi_2 :) : ||\varphi_1 + \varphi_2|| = \xi^{\nu(\varphi_1 + \varphi_2)} \leq gr(\xi^{\nu(\varphi_1)}, \xi^{\nu(\varphi_2)}) \therefore \leq gr(||\varphi_1||, ||\varphi_2||)$.
Thus $\mathfrak{K}_\xi$ is a Hensel-Kürschák field (cf. $D \ 2.4$), denoting by $\mathfrak{K}_\xi$ the field $\mathfrak{K}_\xi$ with the modulus as defined in $D \ 3.5$.

$$Thm \ 3.2 \quad \mathfrak{F} \cdot) \cdot \mathfrak{K}_\xi^H.$$

Hereafter we shall generally denote the elements of $\mathfrak{K}_\xi$ by $h(h_1, h_2, \text{etc.})$. The subscript $\mathfrak{F}$ in $\mathfrak{K}_\xi$ shall be dropped unless it is necessary to bring in evidence the field from which $\mathfrak{K}_\xi$ is formed. The elements of $\mathfrak{F}$ in $\mathfrak{K}_\xi$ (by isomorphism) will, when it is desirable to bring that fact in prominence, be denoted by $f(f_1, f_2, \text{etc.})$.

3.3. Perfection of $\mathfrak{K}_\xi$.—We now prove the theorem:

$$Thm \ 3.3 \quad \mathfrak{F} \cdot) \cdot \mathfrak{K}_\xi^P.$$

Proof: Consider a Cauchy sequence $\{h_n\}$. From property 3' of the modulus either $L_n ||h_n|| = 0$ in which case the limit of the sequence is z or

$$\exists (n_0, \nu) \ni ||h_n|| = \xi^\nu = L_n ||h_n|| \quad (n > n_0)$$

for otherwise if $e < L_n ||h_n||$ which exists, and $a \equiv L_n ||h_n|| - e$,

$$n > n_e :) : \exists (n_1, n_2) \ni ||h_{n_1}|| \neq ||h_{n_2}|| \cdot) \cdot ||h_{n_1} - h_{n_2}|| \\ = gr(||h_{n_1}||, ||h_{n_2}||) > a$$

where n_e is the n_e for the original Cauchy sequence, and this is impossible. With this ν and $h_n \equiv \sum_{i=0}^{\nu} f_{n,i} x^i(n)$, we have

$$\exists n_0 : \exists n > n_0 \cdot (f_{n,i} = f_{n_0,i}) \quad (1)$$

for otherwise $n : \exists (n_1, n_2) > n \ni \|h_{n_1} - h_{n_2}\| \geq \xi^{\nu-1}$, which is not true as $\{h_n\}$ is a Cauchy sequence. Similarly,

$$i : \exists n_0, i : \exists n > n_0, i \cdot (f_{n,\nu+i} = f_{n_0,i,\nu+i}) \quad (2)$$

If we now define $h \equiv \sum_{i=0}^{\infty} f_i x^i$, where $f_i \equiv f_{n_0, i-\nu, i}$ ($i \geq \nu$) in (1) and (2), we see that $i : \|h - h_{n_0,i}\| \leq \xi^{\nu+i}$ and so $h = \lim_{n \rightarrow \infty} Lh_n$.

3.4. Conditions for Compactness of $\mathfrak{X}_\delta$.—Though every $\mathfrak{X}$ as we have seen is perfect, it is not necessarily compact. The necessary and sufficient conditions for compactness are given by the following theorem:

Thm 3.4 $\mathfrak{X}_\delta^{\text{cpt}} \sim \mathfrak{F}^{\text{cpt}} \sim \mathfrak{F}^{\text{finite}}$

Proof: Since $f \neq z \cdot \parallel f \parallel = 1, Lf_n = h \cdot h^n$; therefore if we have a Cauchy sequence of elements of $\mathfrak{F}$ its limit element, when it exists, must be an element of $\mathfrak{F}$. Take now an infinite sequence of elements of $\mathfrak{F}$. 1 is an upper bound of the modulus; therefore by $\mathfrak{X}_\delta^{\text{cpt}}$ this sequence has a Cauchy subsequence with a limit element. This limit element by the above is in $\mathfrak{F}$. Hence,

$$\mathfrak{X}_\delta^{\text{cpt}} \subset \mathfrak{F}^{\text{cpt}}.$$

It is also obvious that $\mathfrak{F}^{\text{finite}} \subset \mathfrak{F}^{\text{cpt}}$, for in this case every infinite sequence of elements of $\mathfrak{F}$ must have at least one of its elements repeated an infinite number of times.

Next suppose $\mathfrak{F}^{\text{cpt}}$; consider $\{h_n\} \ni \bar{B} \|h_n\| < \infty$, say $< \xi^t$. Let $h_n \equiv \sum f_{n,i} x^i(n)$. Then $f_{n,i} = z(n, i < t)$. Since every sequence in $\mathfrak{F}$ has 1 as an upper bound of the modulus, and hence by $\mathfrak{F}^{\text{cpt}}$ has a Cauchy subsequence with a limit element the sequence $\{f_{n,i}\}$ has a Cauchy subsequence with a limit element. Take the corresponding subsequence of $\{h_n\}$. If the coefficients in this sequence be denoted by letters with the upper subscript 1, we have a sequence whose terms begin with a power not less than t and the coefficients of x^t form a Cauchy sequence. Also since every non-zero element of $\mathfrak{F}$ has the modulus 1, after a finite number of terms, the terms in this coefficient sequence are identical. The coefficient sequence $\{f_{n,t+1}^{(1)}\}$ has similarly a Cauchy subsequence. Take the corresponding subsequence of $\{h_n\}$. If we continue in this manner, and denote the successive *distinct* subsequences of $\{h_n\}$ obtained in this manner by $\{h_n^{(1)}\}, \{h_n^{(2)}\}, \dots, \{h_n^{(r)}\}, \dots$, we have

$$\{h_n\} \supset \{h_n^{(r)}\} \supset \{h_n^{(r+1)}\} \quad (r > 0) \quad (1)$$

and

$$r > 0 :) : \exists n_r : \exists : m > r \cdot n > n_r \cdot i < t + r :) : f_{n_r}^{(m)} = f_{n_r, i}. \quad (2)$$

If this series of subsequences has a last term, say $\{h_n^{(r)}\}$, then its coefficient sequences are all Cauchy sequences and it can be easily proved that $\{h_n^{(r)}\}$ is itself a Cauchy sequence, and therefore by $\mathfrak{K}_\delta^p$, with a limit element in $\mathfrak{K}_\delta$.

If the series of subsequences has no last term, take the subsequence of $\{h_n\}$ formed in the following manner: from each term of the infinite series select the first element which does not belong to the next term in the sequence (such elements do exist, since the terms of the infinite series are distinct). By using (2), this sequence can be seen to be a Cauchy sequence and therefore by the perfection of $\mathfrak{K}_\delta$, with a limit element in $\mathfrak{K}_\delta$.

Finally, to prove $\mathfrak{F}^{\text{cpt}} \cdot) \cdot \mathfrak{F}^{\text{finite}}$. We prove this by proving directly the contrapositive, viz.,

$$\mathfrak{F}^{\text{infinite}} \cdot) \cdot \mathfrak{F}^{\text{-cpt}}.$$

For consider $\{f_n\}^{\text{distinct}}$; then $\overline{B}_n \|f_n\| = 1$. If this sequence has a Cauchy subsequence, $e \cdot) \cdot \exists (n_1, n_2) \ni \|f_{n_1} - f_{n_2}\| < e$; but by $\{f_n\}^{\text{distinct}}$ and the fact that $f \neq z \cdot) \cdot \|f\| = 1$, we have $n_1 \neq n_2 \cdot) \cdot \|f_{n_1} - f_{n_2}\| = 1$, and these contradict each other when $e < 1$. Hence the result.

Thus the theorem is completely proved.

Corollary: Since every finite field is of the type* $[p^q]$, where p is prime and q a positive integer, we have:

$$\text{Thm 3.5} \quad \mathfrak{K}_\delta^{\text{cpt}} \cdot \sim \cdot \mathfrak{F}^{\text{type } [p^q]}.$$

3.5. Subfield $R(\mathfrak{F}, x)$ of $\mathfrak{K}_\delta$.—Since rational integral functions of a transcendental x with coefficients in $\mathfrak{F}$ form a subclass of $\mathfrak{K}_\delta$, viz., the class of all finitely non-zero functions on $\mathfrak{F}$ to $\mathfrak{F}$, with z as the functional value for negative values of the argument, the field of all rational functions of such an x is a subfield of $\mathfrak{K}_\delta$. Thus,

$$\text{Thm 3.6} \quad \mathfrak{F} \cdot) \cdot \mathfrak{K}_\delta \supset R(\mathfrak{F}, x).$$

Further,†

$$\text{Thm 3.7} \quad \mathfrak{F}^{\text{finite or of the type } \mathbb{P}^p} \cdot) \cdot \mathfrak{K}_\delta \supsetneq R(\mathfrak{F}, x).$$

Proof: The proof is partly suggested by Hensel's proof of the theorem that the field of the p -adic numbers includes properly the field of ordinary rationals.‡ In analogy with Hensel we proceed to show that under the conditions in the hypotheses every element of $R(\mathfrak{F}, x)$ corresponds to a

* M, p. 220.

† Here the symbol $\supsetneq$ reads "properly includes."

‡ H, p. 38.

periodic element of $\mathfrak{K}_{\delta}$. A periodic element is defined thus:

D 3.6 $(h \equiv \sum f_i x^i)^{\text{periodic}} \equiv h : \exists (s, t > 0) \ni i > s \cdot) \cdot f_i = f_{i+t};$

the smallest such t is called the period.

All polynomials are obviously periodic, the period being 1 and the repeated element z . The product of two periodic elements is easily seen to be periodic. Thus it is sufficient to prove that the reciprocal of a polynomial φ irreducible in $\mathfrak{F}$ is a periodic element.

In case $\mathfrak{F}$ is of the type $\mathfrak{B}^p$, that is, the absolute algebraic field * characteristic p , φ is of the first degree in x . In case $\varphi = fx$, f being a non-zero element of $\mathfrak{F}$, the reciprocal of φ is $f'x^{-1}$, where f' is the reciprocal of f , and is, therefore, obviously periodic. In case $\varphi = fx + f_1$ where none of f, f_1 are zero, its reciprocal is $f'_1/(1 + f_2x)$, where f'_1 is the reciprocal of f_1 and $f = f_1f_2$. But $1/(1 + f_2x)$ equals:

$$1 - f_2x + f_2^2x^2 - \dots + (-1)^r f_2^r x^r + \dots$$

and since f_2 is algebraic with respect to the field $[0, 1, 2, \dots, p-1]$ there exists† a positive integer n such that $f_2^n = 1$. Hence the series is periodic.

In case $\mathfrak{F}$ is finite, $\mathfrak{F}$ is a Galois Field‡ $[p^q]$. If the polynomial φ irreducible in $\mathfrak{F}$ is of degree n in x , the class of all polynomials ψ in x with coefficients in $\mathfrak{F}$ falls modulo φ into p^{nq} classes of congruent polynomials, and hence there exists§ a positive integer m such that $x^m \equiv 1 \pmod{\varphi}$, that is to say, there exists a polynomial ψ such that $\varphi(x)\psi(x) = x^m - 1$. Thus:

$$1/\varphi(x) = \psi(x)/[\varphi(x)\psi(x)] = \psi(x)/(x^m - 1) = -\psi(x)\{1 + x^m + x^{2m} + \dots\}$$

and is therefore obviously periodic.

3.6. **Adjunction of Elements to $\mathfrak{K}_{\delta}$ algebraic to $\mathfrak{F}$.**—We shall now prove a theorem required for the sequel:

Thm 3.8 $\mathfrak{F} \cdot S^{\text{alg } \mathfrak{F}} \cdot \mathfrak{K}_{\delta(S)} = \mathfrak{K}_{\delta}(S),$

where S denotes a system of elements.

Proof: We put the elements of the two fields into isomorphism in this manner: put the systems $\mathfrak{F}$ and S in one into correspondence with those in the other; since S is algebraic with respect to $\mathfrak{F}$, this correspondence will persist under all the field operations like additions, etc. Also since the moduli of these elements in $\mathfrak{F}$ and S are the same in both cases the iso-

* Cf. 1.2; also S, p. 199.

† S., p. 251.

‡ M, p. 220.

§ This can be proved by the usual methods of Galois Field theory (cf. L. E. Dickson, "History of the Theory of Numbers," Washington, D. C., 1919).

morphism so far is complete. Also both $\mathfrak{K}_{\mathfrak{K}(S)}$ and $\mathfrak{K}_{\mathfrak{K}}(S)$ are extensions of $\mathfrak{F}(S)$. If we now identify the element x and its moduli in the two fields, we see that both $\mathfrak{K}_{\mathfrak{K}(S)}$ and $\mathfrak{K}_{\mathfrak{K}}(S)$ are extensions of $\mathfrak{K}_{\mathfrak{K}}$. Now $\mathfrak{K}_{\mathfrak{K}}(S)$ is the smallest extension* of $\mathfrak{K}_{\mathfrak{K}}$ containing the system S . Therefore $\mathfrak{K}_{\mathfrak{K}(S)} \supset \mathfrak{K}_{\mathfrak{K}}(S)$. Further every element of $\mathfrak{K}_{\mathfrak{K}(S)}$ is of the form $\sum s_i x_i$, where the elements s belong to $\mathfrak{F}(S)$, and is thus an element of $\mathfrak{K}_{\mathfrak{K}}(S)$. Thus the theorem is completely proved.

IV. THE PROPERTIES $p, A, P, cpt.$

4.1. **A Complete Existential Theory.**—In the preceding sections we have defined, with reference to Kürschák fields, the four properties p, A, P , and $cpt.$ If the existence or non-existence of these properties in this order is denoted by positive or negative signs in that order, a given Kürschák field will have one of the $2^4 = 16$ characters $(++++), (+++-), (++-+), \dots, (----)$. In this section we develop a *complete existential theory*† of these properties, that is, we prove 2^4 propositions stating for each of the 2^4 combinations $(++++), \dots, (----)$ of these properties whether there exists or does not exist a Kürschák field having the particular combination of the properties. These propositions are tabulated below:

Thm 4.1-4.16:

Cases.	$p.$	$A.$	$P.$	$cpt.$	Consistent.
1.....	+	+	+	+	-
2.....	+	+	+	-	+
3.....	+	+	-	+	-
4.....	+	+	-	-	+
5.....	+	-	+	+	+
6.....	+	-	+	-	+
7.....	+	-	-	+	-
8.....	+	-	-	-	+
9.....	-	+	+	+	+
10.....	-	+	+	-	+
11.....	-	+	-	+	-
12.....	-	+	-	-	+
13.....	-	-	+	+	+
14.....	-	-	+	-	+
15.....	-	-	-	+	-
16.....	-	-	-	-	+

Note: The + and - entries in the last column denote the existence and non-existence respectively of a Kürschák field with the character denoted by the entries under the first four columns.

Also in looking the entries under p , it is necessary to remember that the entries in the final column do not depend upon any special choice of the

* S, p. 186.

† E. H. Moore, "Introduction to a Form of General Analysis," 1910, Yale University Press, p. 82.

prime p . Thus the entries with $+$ sign under p we interpret as $p \cdot) \cdot \exists \mathfrak{R}^p$, etc., and those with $-$ sign as $\exists \mathfrak{R}^0$, etc.

4.2. Proofs:

(a) 3, 7, 11, and 15 follow from theorem 2.1, viz., $\mathfrak{R}^{\text{cpt}} \cdot) \cdot \mathfrak{R}^P$.

(b) 1 follows from theorem 2.4, viz., $\mathfrak{R}^{A \cdot \text{cpt}} \cdot) \cdot \mathfrak{R}^0$.

(c) The following number systems in analysis with the absolute value as modulus prove 9, 12, 13, and 16 respectively:

- (9) complex numbers,
- (12) algebraic numbers,
- (13) real numbers,
- (16) rational numbers.

(d) To prove 2: consider $(\mathfrak{X}_{\mathfrak{p}^p})_{(AP)}$; this has the properties p, A, P . By theorem 3.4, $\mathfrak{X}_{\mathfrak{p}^p}$ is P and $-cpt$. Using 2.15 we see that $(\mathfrak{X}_{\mathfrak{p}^p})_{(AP)}$ is $-cpt$.

(e) To prove 5: consider $\mathfrak{X}_{[p]}$. This field is $-A$, for the equation $y^2 - x = 0$ has no solution. However by 3.4 it is cpt .

(f) We get 6, an example being $\mathfrak{X}_{\mathfrak{p}^p}$. The proof is similar to the above.

(g) Similarly $\mathfrak{X}_R$ where $R \equiv [\text{all ordinary rationals}]$ gives us 14.

4.3. Proof of 10: We proceed to prove that:

$$Thm\ 4.10_0 \quad \mathfrak{S}_{(AP)}^{-\text{cpt}},$$

where $\mathfrak{S}$ is the field of Hensel's p -adic numbers.

Take an element of $\mathfrak{S}$ whose modulus a is > 1 . Take an infinite sequence of positive distinct ordinary rationals $\{r_n\}$ so that they all lie between two positive rationals r, r' . Let $r_n \equiv l_n/m_n$, where l_n, m_n are positive and relatively prime. Consider the sequence $\{h_n\}$ where $h_n^{m_n} - h_n^{l_n} = 0 \ (n)$. Then $\|h_n\| = a^{r_n} \ (n) \ \therefore \{h_n\}^{\text{distinct}}$. This sequence an upper bound of whose modulus is $gr(a^r, a^{r'})$, cannot have any Cauchy subsequence.

For otherwise,

$$e :) : \exists n_e \exists (n_1, n_2) > n_e \cdot) \cdot \|h_{n_1} - h_{n_2}\| < e,$$

but we have

$$n_1 \neq n_2 :) : \|h_{n_1}\| \neq \|h_{n_2}\| :) : \|h_{n_1} - h_{n_2}\| = gr(\|h_{n_1}\|, \|h_{n_2}\|) \\ \therefore \geq a^r,$$

and taking $e < a^r$, we get a contradiction.

Hence the theorem.

4.4. Proof of 4: Consider $(\mathfrak{X}_{[p]})_A$; $\mathfrak{X}_{[p]}$ is perfect by 3.3. If we now prove that algebraic closure is in this case obtained by the adjunction of an infinite number of elements, then, by 2.21 $(\mathfrak{X}_{[p]})_A^{-P}$ and hence by 2.1

— *cpt.* Now $(\mathfrak{X}_{[p]})_A$ contains $\mathfrak{X}_{\mathfrak{P}^p}$ which by theorem 3.8 can be regarded as obtained from $\mathfrak{X}_{[p]}$ by adjoining algebraic elements to $[p]$. The number of these is, however, infinite; otherwise $\mathfrak{P}^p$ would have only a finite number of elements. Thus $(\mathfrak{X}_{[p]})_A$ is infinite with respect to $\mathfrak{X}_{[p]}$ and so the theorem is proved.

4.5. *Proof of 8:* Ostrowski in his theses* has proved the following theorem:

Thm 4.8₀ $\mathfrak{H}^P \cdot \text{Sais}_1 \mathfrak{H}; \text{infinite} \cdot) \cdot \mathfrak{H}(S)^{-P}.$

We use this theorem to prove 4.8.

Consider $\mathfrak{X}_{[p]}$, $p > 2$; let r be any one of the integers: $2, 3, \dots, p-1$. From Steinitz's paper,† we have

$$q^{\text{prime.} > p :) : (y^q + r)^{\text{irreducible in } [p]. \text{ 1st kind.}}$$

Consider a sequence of primes, $\{q_n\}$ such that

- (1) $q_1 > p$
- (2) $q_n > p^{q_1 q_2 q_3 \dots q_{n-1}}$

and the corresponding infinite system of irreducible polynomials

$$S \equiv (y^{q_n} + r \mid n).$$

Consider now $\mathfrak{X}_{[p]}(S)$. By theorem 3.8, $\mathfrak{X}_{[p]}(S) = \mathfrak{X}_{[p](S)}$. Also S is of the first kind, further if $S_m \equiv (y^{q_n} + r \mid n = 1, 2, \dots, m-1)$, $[p](S_m)$ contains elements whose order is at most $p^{q_1 q_2 \dots q_{m-1}}$ and hence the polynomial $y^{q_m} + r$ is irreducible in $[p](S_m)$ since the order of its roots is $q_m >$ the highest order in $[p](S_m)$. Thus S is a progressive‡ system. We have further $\mathfrak{X}_{[p]}(S)^{-A}$, since the polynomial $y^2 + x$ for instance has still no root. Thus $\mathfrak{X}_{[p]}(S)^{p-A-P-\text{cpt.}}$

* *O*₁, p. 280.

† *S*, p. 231.

‡ *S*, p. 271.

VITA.

I, Vishnu Dattatreya Gokhale, was born in Poona City, India, on the twenty-fourth of February, 1892. I got my secondary and high school education at the Nutan Marathi Vidyalaya, Poona City. I entered Fergusson College, Poona City, in 1907, and did all my undergraduate and part of my graduate work there. My instructors in mathematics were Prin. R. P. Paranjpye and Professor V. B. Naik. I got the degree of Bachelor of Arts in 1911, and Master of Arts 1913, at the University of Bombay. During 1914-1917 I worked as Professor of Mathematics at the Fergusson College, Poona City.

I came to the United States in December, 1917. During 1918-19 I took courses at the University of California under Professor Cajori, and Drs. McDonald, Buck, Sperry and Bernstein. I joined the University of Chicago in autumn, 1919, and during nine quarters took courses in mathematics under Professors Moore, Bliss, Dickson and Wilczynski, the major part of my work being under Professor Moore.

This paper was written under the direction of Professor Moore and I wish to express my appreciation of his constant encouragement and invaluable criticism while engaged in the investigation.

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